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COEFFICIENT ESTIMATES FOR DIRICHLET SERIES

BY
W. T. SLEDD(1)

1. **Introduction.** The primary purpose of this paper is to study coefficient estimates of Dirichlet series

$$f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x), \qquad \lambda_n \to \infty, \quad 0 < \lambda_1 < \lambda_2 < \cdots,$$

for which the sequence $\{\lambda_n\}$ satisfies certain lacunarity conditions. D. Gaier [2] has used such estimates in one of his proofs of the high-indices theorem for Borel summability, in which results on the order of an entire function are inferred from its rate of growth along the real line. A by-product of his work is the estimate $|a_n| \leqq 2\lambda_n p_n \|f\|_1$ when $\sum \lambda_n^{-1} < \infty$, $f \in L^1(0, \infty)$, and

$$p_n = \prod_{k \neq n} \left| \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right|.$$

Some subsequent papers in much the same vein are due to G. Halász [3] and to J. M. Anderson and K. G. Binmore [1]. Halász has obtained estimates for a_n, $\sum_1^N a_n$, and $\sum_1^N |a_n|$ when $\lambda_{n+1}/\lambda_n \geqq q > 1$ and f satisfies appropriate conditions. The paper of Anderson and Binmore is concerned with the application of coefficient estimates to the study of entire functions. Their work includes the estimate $|a_n| \leqq (2\lambda_n)^{1/2} p_n \|f\|_2$ where p_n is as above, $\sum \lambda_n^{-1} < \infty$, and $f \in L^2(0, \infty)$.

L. Schwartz [8] had earlier obtained results which essentially include both the estimates of Gaier and of Anderson and Binmore. Indeed, as the referee has pointed out, S. Mandelbrojt [5] had also established inequalities of the same kind and these appear in larger form in [6]. Later, Mandelbrojt's results were extended by F. Sunyer Belaguer [9], [10].

There is some question left to the sharpness of Schwartz's estimates and in §2 it is proved that if $1 \leqq p \leqq 2$ and $f \in L^p(0, \infty)$ then

$$|a_n| \leqq (2\lambda_n)^{1/p} p_n \|f\|_p.$$

I do not know if the constant is sharp when $p < 2$, or if the order of the estimate still holds when $p > 2$ but in §2 there are estimates when $p > 2$ for functions which are subject to more stringent conditions. Also in §2 there are estimates for the sequence of partial sums.

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These results are applied in §3 to obtain some gap theorems. W. Rudin [7] showed that if $\liminf \lambda_{n+1}/\lambda_n = 1$ then there is a Dirichlet series so that $f' \in L^1(0, \infty)$ yet $\sum a_n$ diverges. In §3 it is shown that there are sequences $\{\lambda_n\}$ so that $\lim \lambda_{n+1}/\lambda_n = 1$, but so that if $f' \in L^p(0, \infty)$ when $p > 1$, or if $f'' \in L^1(0, \infty)$, then $\sum |a_n| < \infty$.

2. Coefficient estimates.

THEOREM 2.1. *If $f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x > 0$, $\sum \lambda_n^{-1} < \infty$, and $f \in L^p(0, \infty)$, $1 \leqq p \leqq 2$, then $|a_n| \leqq (2\lambda_n)^{1/p} p_n \|f\|_p$ where*

$$p_n = \prod_{k \neq n} \left| \frac{\lambda_k + \lambda_n}{\lambda_k - \lambda_n} \right|.$$

Proof. Let

$$f_N(x) = \sum_{k=1}^{N} a_k \exp(-\lambda_k x)$$

and

$$F(z) = \sum_{k=1}^{N} \frac{a_k \exp(-\lambda_k \varepsilon)}{\lambda_k - z}, \qquad \varepsilon > 0.$$

Then if $\operatorname{Re} z < 0$

$$F(z) = \int_0^\infty e^{zt} f_N(t+\varepsilon)\, dt.$$

Let

$$B(z) = \prod_{k=1}^{N} \frac{\lambda_k - z}{\lambda_k + z}$$

so that

$$G(z) = F(z)B(z) = \sum_{k=1}^{N} \frac{a_k \exp(-\lambda_k \varepsilon)}{z + \lambda_k} B_k(z)$$

where

$$B_k(z) = \left(\prod_{j \neq k}^{N} \frac{\lambda_j - z}{\lambda_j + z} \right).$$

Now $B_k(z)/(z+\lambda_k) \in H^2$, a Hardy class of functions in the right half-plane, and so the same is true of G. Consequently, G may be represented by the Poisson integral of its values on the imaginary axis, and

$$\|G(x+iy)\|_q \leqq \|G(iy)\|_q$$

if $x > 0$ and $q \geqq 1$ [4, p. 124, and p. 128].

Next, let Γ be the boundary of a rectangle in the right half-plane with sides parallel to the real and imaginary axes, and enclosing λ_n.

Let $\zeta^{2/p} = \exp(2/p \log \zeta)$ for $|\arg \zeta| < \pi/2$, with $\log 1 = 0$. Then by the residue theorem

$$\frac{a_n \exp(-\lambda_n \varepsilon) B_n(\lambda_n)}{(2\lambda_n)^{2/p}} = \frac{-1}{2\pi i} \int_\Gamma \frac{F(z) B_n(z)}{(z+\lambda_n)^{2/p}}\, dz.$$

But since $G(z)$ goes to zero uniformly in any fixed half-plane $\operatorname{Re} z \geqq \delta > 0$ [4, p. 125] it follows that

$$\frac{a_n \exp(-\lambda_n \varepsilon) B_n(\lambda_n)}{(2\lambda_n)^{2/p}} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x_n' + iy) B_n(x_n' + iy)}{(x_n' + iy + \lambda_n)^{2/p}} \, dy$$

$$- \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{F(x_n'' + iy) B_n(x_n'' + iy)}{(x_n'' + iy + \lambda_n)^{2/p}} \, dy$$

$$= I_1 + I_2$$

where $x_n' < \lambda_n < x_n''$.

Then

$$|I_1| \leqq \frac{1}{2\pi} \|G(x_n' + iy)\|_q \left[\int_{-\infty}^{\infty} \left| \frac{x_n' + iy + \lambda_n}{x_n' + iy - \lambda_n} \right|^p \frac{dy}{|x_n' + iy + \lambda_n|^2} \right]^{1/p}$$

where $1/p + 1/q = 1$. A similar estimate holds for I_2. Since $\|G(x_n'' + iy)\|_q \leqq \|G(iy)\|_q$ we may let $x_n'' \to \infty$ and $x_n' \to 0$ to conclude that

$$\frac{|B_n(\lambda_n) a_n \exp(-\lambda_n \varepsilon)|}{(2\lambda_n)^{2/p}} \leqq \frac{1}{2\pi} \|G(iy)\|_q \left(\int_{-\infty}^{\infty} \frac{dy}{y^2 + \lambda_n^2} \right)^{1/p}$$

$$= \frac{1}{2\pi} \left(\frac{\pi}{\lambda_n} \right)^{1/p} \|G(iy)\|_q.$$

Moreover, since $|B(iy)| = 1$, then $\|G(iy)\|_q = \|F(iy)\|_q$, but $F(iy)$ is the Fourier transform of $f_N(t+\varepsilon)$, so that by the Hausdorff-Young inequality [11, p. 96], $\|G(iy)\|_q \leqq (2\pi)^{1/q} \|f_N(t+\varepsilon)\|_p$.

Combining these inequalities gives

$$|a_n \exp(-\lambda_n \varepsilon)| \leqq \frac{(2\lambda_n)^{1/p}}{|B_n(\lambda_n)|} \|f_N(t+\varepsilon)\|_p.$$

If $\varepsilon > 0$, summation by parts shows that there is a constant C such that

$$|f_N(t+\varepsilon)| \leqq C \exp(-\lambda_1 t)$$

and

$$|f(t+\varepsilon)| \leqq C \exp(-\lambda_1 t).$$

Since $f_N(t) \to f(t)$ when $t > 0$, the Lebesgue dominated convergence theorem implies that

$$\|f_N(t+\varepsilon) - f(t+\varepsilon)\|_p \to 0.$$

But $\|f(t+\varepsilon)\|_p \leqq \|f\|_p$ and so letting $N \to \infty$ and then $\varepsilon \to 0$ in the above inequality completes the proof.

The next theorem deals with the case where $p > 2$. The technique is essentially that of Halász [3].

THEOREM 2.2. *If*

(a) $f(z)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n z)$ *converges when* $\operatorname{Re} z>0$;

(b) $\|f(re^{i\theta})\|_p \leqq M$ *whenever* $|\theta| \leqq \eta = \delta\pi/2 < \pi/2$;

(c) $\sum \lambda_n^{-1/\beta} < \infty$, *where* $\beta+\delta=\gamma>1$, *and* $\beta<2$; *then*

$$|a_n| \leqq \frac{2M\beta\lambda_n^{1/p}}{|q\cos(\gamma\pi/2)|^{1/q}} \prod_{k\neq n} \left|\frac{\lambda_k^{1/\beta}+\lambda_n^{1/\beta}}{\lambda_k^{1/\beta}-\lambda_n^{1/\beta}}\right|$$

where $1/p+1/q=1$.

Proof. Let

$$\phi_N(z) = \sum_{k=1}^{N} a_k \exp(-\lambda_k z)$$

and

$$F(z) = \sum_{k=1}^{N} \frac{a_k \exp(-\lambda_k \varepsilon)}{\lambda_k - z}, \qquad \varepsilon > 0.$$

Then if $\operatorname{Re} ze^{i\alpha}<0$, and $|\alpha|<\pi/2$

$$F(z) = \int_0^\infty \exp(zte^{i\alpha})\phi_n(te^{i\alpha}+\varepsilon)e^{i\alpha}\,dt.$$

But if $z=re^{i\phi}$ and $\operatorname{Re} ze^{i\alpha}<0$, then

$$|F(z)| \leqq \left[\int_0^\infty \exp(rtq\cos(\theta+\alpha))\,dt\right]^{1/q} \|\phi_N(te^{i\alpha}+\varepsilon)\|_p$$

$$= \|\phi_N(te^{i\alpha}+\varepsilon)\|_p \frac{1}{|qr\cos(\theta+\alpha)|^{1/q}}.$$

Now let $(\zeta)^{1/\beta}$ be defined for $|\arg\zeta|<\pi$, $(1)^{1/\beta}=1$, and set

$$B(z) = \prod_1^N \frac{1-(z/\lambda_k)^{1/\beta}}{1+(z/\lambda_k)^{1/\beta}}.$$

Similarly define $(\zeta)^{1/q}$, so that letting

$$\alpha = \pm\eta, \qquad z = r\exp(\pm i\pi\beta/2),$$

$$|z^{1/q}F(z)B(z)| \leqq \frac{\|\phi_N(te^{i\alpha}+\varepsilon)\|_p}{|q\cos(\gamma\pi/2)|^{1/q}} = A.$$

Now $z^{1/q}F(z)B(z)$ is of relatively slow growth for $|\arg z|<\beta\pi/2$, so that by the Phrägmen-Lindelöf theorems [12, p. 180]

$$|z^{1/q}F(z)B(z)| \leqq A, \qquad |\arg z| \leqq \beta\pi/2.$$

In particular, if $z=\lambda_n$ then

$$\frac{\lambda_n^{1/q}|a_n|\exp(-\lambda_n\varepsilon)}{2\beta\lambda_n} \prod_{k\neq n} \left|\frac{\lambda_k^{1/\beta}-\lambda_n^{1/\beta}}{\lambda_k^{1/\beta}+\lambda_n^{1/\beta}}\right| \leqq A.$$

As in the proof of Theorem 2.1, let $N\to\infty$ and then $\varepsilon\to 0$ to obtain the desired inequality.

The next theorem deals with sequences of partial sums.

THEOREM 2.3. *If* $f(x)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ *converges for each* $x>0$, $\sum \lambda_n^{-1}<\infty$ *and* $f' \in L^p(0,\infty)$, $1<p\leqq 2$, *then*

$$\left|\sum_{k=r}^{s} a_k\right| \leqq A_p\|f'\|_p[C_{r-1}+C_s]$$

where A_p *is a constant depending on p alone, and*

$$C_n = \operatorname*{Min}_{\lambda_n<x<\lambda_{n+1}} \frac{1}{x^{1-1/p}} \prod_{k=1}^{\infty}\left|\frac{x+\lambda_k}{x-\lambda_k}\right|.$$

Proof. The proof is very much the same as that of Theorem 2.1 and hence many of the details are omitted.

Let

$$\phi_N(x) = \sum_{k=1}^{N} \lambda_k a_k \exp(-\lambda_k x),$$

$$F(z) = \sum_{k=1}^{N} \frac{\lambda_k a_k \exp(-\lambda_k(z+\varepsilon))}{\lambda_k - z}, \qquad \varepsilon > 0,$$

and

$$B(z) = \prod_{k=1}^{N} \frac{\lambda_k - z}{\lambda_k + z}.$$

If Γ is a rectangle in the right half-plane enclosing $\lambda_r, \ldots, \lambda_s$ and no others then

$$\begin{aligned}\sum_{k=r}^{s} a_k \exp(-\lambda_k\varepsilon) &= -\frac{1}{2\pi i}\int \frac{F(z)}{z}\,dz\\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(x_r+iy)}{x_r+iy}\,dy - \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{F(x_s+iy)}{x_s+iy}\,dy\\ &= I_1+I_2,\end{aligned}$$

where $\lambda_{r-1}<x_r<\lambda_r \leqq \lambda_s < x_s < \lambda_{s+1}$.

If $G(z)=F(z)B(z)$ then

$$|I_1| \leqq \frac{1}{2\pi}\|G(x_r+iy)\|_q D_r\left(\int_{-\infty}^{\infty}\frac{dy}{|x_r+iy|^p}\right)^{1/p}$$

where $1/p+1/q=1$ and

$$D_r = \max_y \frac{1}{|B(x_r+iy)|}.$$

A similar estimate holds for I_2 and since

$$\|G(x_r+iy)\|_q \leqq \|G(iy)\|_q = \|F(iy)\|_q \leqq (2\pi)^{1/q}\|\phi_N(t+\varepsilon)\|_p$$

it follows that

$$\left|\sum_{k=r}^{s} a_k \exp(-\lambda_k\varepsilon)\right| \leqq A_p\|\phi_N(t+\varepsilon)\|_p\left(\frac{D_r}{x_r^q}+\frac{D_s}{x_s^q}\right).$$

Letting $N\to\infty$ and then $\varepsilon\to 0$ completes the proof.

THEOREM 2.4. *If* $f(x)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ *converges for each* $x>0$, $\sum \lambda_n^{-1}<\infty$, *and* $f'' \in L^1(0,\infty)$ *then*

$$\left|\sum_{k=r}^{s} a_k\right| \leqq \frac{\|f''\|}{2}\left[\operatorname*{Min}_{\lambda_{r-1}<x<\lambda_r} \frac{1}{x}\prod_{k=1}^{\infty}\left|\frac{x+\lambda_k}{x-\lambda_k}\right| + \operatorname*{Min}_{\lambda_s<x<\lambda_{s+1}} \frac{1}{x}\prod_{k=1}^{\infty}\left|\frac{x+\lambda_k}{x-\lambda_k}\right|\right].$$

Proof. The proof is very similar to the proof of Theorem 2.3 and so will be omitted.

3. **Some applications.** Before applying the estimates of §2 it is necessary to establish a result about the special sequence $\lambda_n=\exp(n^\alpha)$.

THEOREM 3.1. *If* $\lambda_n=\exp(n^\alpha)$ *and* $1>\alpha>0$ *then*

$$P_n = \prod_{k\neq n}\left|\frac{\lambda_k+\lambda_n}{\lambda_k-\lambda_n}\right| \leqq \exp Cn^{2(1-\alpha)}$$

where C is a constant, dependent only on α.

Proof. Since $\lambda_k/\lambda_{k-1} \geqq \lambda_{k+1}/\lambda_k$, it follows that

$$\sum_{k=1}^{n-1}\frac{\lambda_k}{\lambda_n-\lambda_k} = \sum_{k=1}^{n-1}\frac{\lambda_k}{\lambda_n}\sum_{j=0}^{\infty}\left(\frac{\lambda_k}{\lambda_n}\right)^j = \sum_{j=1}^{\infty}\frac{1}{\lambda_n^j}\sum_{k=1}^{n-1}\lambda_k^j$$

$$\leqq \sum_{j=1}^{\infty}\frac{1}{\lambda_n^j}\lambda_{n-1}^{j-1}\sum_{k=1}^{n-1}\lambda_k \leqq \sum_{j=1}^{\infty}\left(\frac{\lambda_{n-1}}{\lambda_n}\right)^j\frac{1}{1-\lambda_{n-2}/\lambda_{n-1}}$$

$$\leqq \frac{1}{1-\lambda_{n-1}/\lambda_n}\,\frac{1}{1-\lambda_{n-2}/\lambda_{n-1}} \sim \frac{1}{\alpha^2 n^{2(\alpha-1)}}.$$

And

$$\sum_{k=n+1}^{\infty}\frac{\lambda_n}{\lambda_k-\lambda_n} = \sum_{j=1}^{\infty}\lambda_n^j\sum_{k=n+1}^{\infty}\frac{1}{\lambda_k^j}.$$

But

$$\sum_{k=n+1}^{\infty}\frac{1}{\lambda_k^j} \leqq \frac{1}{\lambda_{n+1}^j}+\int_{n+1}^{\infty}\exp(-jx^\alpha)\,dx$$

so that

$$\sum_{j=1}^{\infty}\lambda_n^j\sum_{k=n+1}^{\infty}\frac{1}{\lambda_k^j} \leqq \sum_{j=1}^{\infty}\left(\frac{\lambda_n}{\lambda_{n+1}}\right)^j+\int_{n+1}^{\infty}\sum_{j=1}^{\infty}\left(\frac{\lambda_n}{\exp(x^\alpha)}\right)^j dx$$

$$\leqq \frac{\lambda_n}{\lambda_{n+1}}\,\frac{1}{1-\lambda_n/\lambda_{n+1}}+\frac{1}{\alpha}\int_{(n+1)^\alpha}^{\infty}\frac{\lambda_n e^{-u}}{u^{1-1/\alpha}}\,\frac{du}{1-\lambda_n e^{-u}}$$

$$\leqq \frac{\lambda_n}{\lambda_{n+1}}\,\frac{1}{1-\lambda_n/\lambda_{n+1}}+\frac{1}{\alpha}\,\frac{\lambda_n}{1-\lambda_n/\lambda_{n+1}}\int_{(n+1)^\alpha}^{\infty}u^{1/\alpha-1}e^{-u}\,du.$$

But

$$\int_{(n+1)^\alpha}^{\infty}u^{1/\alpha-1}e^{-u}\,du = (n+1)^{1-\alpha}\lambda_{n+1}^{-1}+\frac{1-\alpha}{\alpha}\int_{(n+1)^\alpha}^{\infty}\frac{e^{-u}u^{1/\alpha-1}}{u}\,du$$

$$\leqq (n+1)^{1-\alpha}\lambda_{n+1}^{-1}+\frac{1-\alpha}{\alpha}\,\frac{1}{(n+1)^\alpha}\int_{(n+1)^\alpha}^{\infty}e^{-u}u^{1/\alpha-1}\,du,$$

so for large values of n

$$\sum_{k=n+1}^{\infty} \frac{1}{\lambda_k^{j}} \leqq \frac{1}{1-\lambda_n/\lambda_{n+1}} + \frac{1}{\alpha} \frac{(n+1)^{1-\alpha}}{1-\lambda_n/\lambda_{n+1}} \left(1-\left(\frac{1-\alpha}{\alpha(n+1)^{\alpha}}\right)\right)^{-1}$$

$$\sim \frac{1}{\alpha n^{\alpha-1}} + \frac{1}{\alpha^2} \frac{1}{(n+1)^{2(\alpha-1)}} \leqq \frac{2}{\alpha^2} \frac{1}{(n+1)^{2(\alpha-1)}}.$$

So

$$\begin{aligned} p_n &= \prod_{k=1}^{n-1} \frac{\lambda_k+\lambda_n}{\lambda_n-\lambda_k} \prod_{k=n+1}^{\infty} \frac{\lambda_k+\lambda_n}{\lambda_k-\lambda_n} \\ &\leqq \exp 2 \sum_{k=1}^{n-1} \frac{\lambda_k}{\lambda_n-\lambda_k} \cdot \exp 2 \sum_{k=n+1}^{\infty} \frac{\lambda_n}{\lambda_k-\lambda_n} \\ &\leqq \exp Cn^{2(1-\alpha)} \end{aligned}$$

where C is some constant.

With this estimate the following theorems may be proved.

THEOREM 3.2. *If $p>1$ then there is a sequence $\{\lambda_n\}$ such that if*

$$f(x) = \sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$$

converges for each $x>0$ and $f' \in L^p(0,\infty)$ then $\sum |a_n|^{\varepsilon} < \infty$ for each $\varepsilon > 0$.

Proof. Assume without loss of generality that $1<p\leqq 2$, for if $p>2$ and $f' \in L^p(0,\infty)$ then $f' \in L^r(0,\infty)$ for each $r<p$. This is true since the integrability of f' depends only on its integrability over (0, 1).

Now by Theorem 2.1 and Theorem 3.1, if $\lambda_n = \exp(n^{\alpha})$ and $2/3<\alpha<1$ then

$$|a_n|^{\varepsilon} = O(\exp(\varepsilon C n^{2(1-\alpha)} - \varepsilon q^{-1} n^{\alpha})),$$

and this estimate shows that $\sum |a_n|^{\varepsilon}$ converges.

THEOREM 3.3. *There exists a sequence $\{\lambda_n\}$ such that if $f(x)=\sum_{n=1}^{\infty} a_n \exp(-\lambda_n x)$ converges for each $x>0$ and $f'' \in L^1(0,\infty)$ then $\sum |a_n|^{\varepsilon}$ converges.*

Proof. This follows from Theorem 2.4. The proof is similar to that of Theorem 3.2.

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MICHIGAN STATE UNIVERSITY,
EAST LANSING, MICHIGAN 48823
WESTFIELD COLLEGE, UNIVERSITY OF LONDON,
LONDON, ENGLAND